Math 138 – Midterm

October 22, 2025

Instructions:

- You have 2 hours to complete this exam.
- No external resources are allowed.
- Do not hesitate to ask for clarification on exam questions.

Question 1. (15 pts)

Spot the error in the following proof, and explain why it is wrong.

Proposition. *Every horse is the same color.*

Proof. We prove by induction that every collection S of n horses has the property that all horses in S have the same color, which clearly implies the proposition. When $n = 1$ this is trivially true as S only contains one horse. Suppose the claim is true for S of size n, and let T be a set of horses of size $n + 1$. Choose any horse h in T, and consider the set $T - \{h\}$. This is a set of n horses, so by the induction hypothesis all horses in $T - \{h\}$ have the same color. Let h' be a different horse in T and consider $T - \{h'\}$. Again by induction all horses in $T - \{h'\}$ have the same color. Let h'' be any horse different than h and h'. Then, as both h' and h'' belong to $T - \{h\}$ they have the same color, and as both h and h'' both belong to $T - \{h'\}$ they both have the same color. So, h, h', and h'' all have the same color. As h, h', and h'' were arbitrary we deduce that all horses in T have the same color as desired. □

Solution: The issue comes in the lack of sufficient base cases. Namely, the base case when $n = 1$ is correct, and the argument for the inductive hypothesis is argued correctly, but only assuming that there are 3 distinct horses h, h', and h''. Because of this, one needs base cases for all $n < 3$ and so, in particular, also for $n = 2$. But, this base case fails as one can have two horses of a different color. ■

Rubric:

- **(5 pts)** Coherence of explanation.
- **(5 pts)** Observing that the issue is in base cases.
- **(5 pts)** Explaining what the precise issue is.

Question 2. (15 pts)

Prove that for all $n \geqslant 0$, the equality

$$F_0 + F_1 + F_2 + \cdots + F_n = F_{n+2} - 1$$

Solution: We proceed by induction.

Base case: When $n = 0$ this equality reduces to $F_0 = F_2 - 1$, which is true as $F_0 = 0$ and $F_2 = 1$.

Inductive hypothesis: Let us assume the equality holds true for some $n \geqslant 0$. Then, we wish to prove the claim for $n + 1$ or, in other words that

$$F_0 + F_1 + \cdots + F_n + F_{n+1} = F_{(n+1)+2} - 1. \tag{1}$$

But, the left-hand side can be rewritten as

$$\big(F_0 + F_1 + \cdots + F_n\big) + F_{n+1},$$

and by the inductive hypothesis this bracketed expression is equal to $F_{n+2} - 1$. So then, we can rewrite this as

$$F_{n+2} - 1 + F_{n+1}.$$

But, by definition, $F_{n+3} = F_{n+2} + F_{n+1}$, and so we can further rewrite this as

$$F_{n+3} - 1,$$

which is the right-hand side of (1), as desired. ■

Rubric:

- **(5 pts)** Coherence in proof writing.
- **(2 pts)** Giving a correct proof strategy (e.g., induction).
- **(2 pts)** Correctly verifying the base case.
- **(6 pts)** Correctly verifying the induction hypothesis.

Question 3. (20 pts)

Suppose that $p \neq q$ are primes. Show that $\sqrt[3]{pq^2}$ is not rational.

Solution: Write $x = pq^2$, a rational number. Then, from class we know that $\sqrt[3]{x}$ belongs to $\mathbb{Q}$ if and only if

1. $\operatorname{sgn}(x)$ has a cuberoot in $\mathbb{Q}$,
2. for all primes ℓ one has that $v_\ell(x)$ is divisible by 3.

That said, as p and q are distinct primes, we see that $v_p(x) = v_p(pq^2) = 1$ which is not divisible by 3. So, $\sqrt[3]{x} \notin \mathbb{Q}$ as desired. ■

Rubric:

- **(7 pts)** Coherence in proof writing.
- **(3 pts)** Giving a correct strategy (e.g., using p-adic valuation or "Assume $\sqrt[3]{pq^2} = \frac{a}{b} \ldots$"
- **(3 pts)** Using the Fundamemental Theorem of Algebra.
- **(7 pts)** Sucessfully executing idea.

Question 4. 20 pts

Let A and B be subsets of a set S. Prove that

$$(A \cup B) \triangle (A \cap B) = A \triangle B.$$

Solution: Assume first that x belongs to $(A \cup B) \triangle (A \cap B)$. Then, by definition, either

1. $x \in A \cup B$ and $x \notin A \cap B$, or
2. $x \in A \cap B$ and $x \notin A \cup B$.

In the first case, we see that either $x \in A$ or $x \in B$. If $x \in A$, then as $x \notin A \cap B$ we see that $x \notin B$, and so $x \in A - B \subseteq A \triangle B$. If in the first case $x \in B$, a symmetric argument shows $x \in B - A \subseteq A \triangle B$. But, note that the second case cannot happen as $A \cap B \subseteq A \cup B$. So, in any case we have shown $(A \cup B) \triangle (A \cap B) \subseteq A \triangle B$.

Conversely, suppose that $x \in A \triangle B$. Then, either

1. $x \in A$ and $x \notin B$,
2. $x \in B$ and $x \notin A$.

In the first case as $x \in A$ we have $x \in A \cup B$ and as $x \notin B$ we have $x \notin A \cap B$. Thus, $x \in (A \cup B) \triangle (A \cap B)$. Case two is argued symmetrically, and so in any case we have $A \triangle B \subseteq (A \cup B) \triangle (A \cap B)$ as desired. ■

Rubric:

- **(6 pts)** Coherence in proof writing.
- **(4 pts)** Proof strategy.
- **(5 pts)** Left-hand side contained in right-hand side shown directly.
- **(5 pts)** Right-hand side contained in left-hand side shown directly.

Question 5. (30 pts)

1. **(20 pts)**

   Prove by induction that the number of subsets of $\{1, \ldots, n\}$ is 2^n. (*Hint: prove that if S is the collection of subsets of $\{1, \ldots, n+1\}$ containing $n+1$, and T is the collection not containing $n+1$, then S and T are in 1-to-1 correspondence.*)

2. **(10 pts)**

   Use 1. to show that

   $$2^n = \sum_{k=0}^{n} \binom{n}{k}.$$

   (*Note: you can use the claim from 1. to solve 2. even if you didn't solve 1.*)

Solution:

1. We proceed by induction.

   Base case: When $n = 0$ we have that $\{1, \ldots, n\}$ is empty, and there is precisely $1 = 2^0$ subsets of $\varnothing$: namely $\varnothing$ itself.

   Inductive hypothesis: Assume the claim is true for $n \geqslant 0$. We show the claim is true for $n+1$. To see this. observe that we can decompose the collection of $\{1, \ldots, n, n+1\}$ into two disjoint subcollections S and T

   (a) S consisting of those subsets of $\{1, \ldots, n, n+1\}$ containing $n+1$,

   (b) T consists of those subsets of $\{1, \ldots, n, n+1\}$ not containing $n+1$.

   Note that S has the same size as the collection of all subsets of $\{1, \ldots, n\}$: namely for a set $X \in S$ one has that $X - \{n+1\}$ is a subset of $\{1, \ldots, n\}$, and for a subset $Y \subseteq \{1, \ldots, n\}$ one has that $Y \cup \{n+1\} \in S$—this is a 1-to-1 correspondence.

   Similarly, note that T also has the same size as the collection of all subsets of $\{1, \ldots, n\}$—in fact, it is equal to this set as an element of T is just a subset of

   $$\{1, \ldots, n, n+1\} - \{n+1\} = \{1, \ldots, n\}.$$

   By the inductive hypothesis we have that S and T both have size 2^n, and has the collection of all subsets of $\{1, \ldots, n, n+1\}$ has size the sum of the sizes of S and T, we deduce there are $2^n + 2^n = 2^{n+1}$ subsets of $\{1, \ldots, n, n+1\}$ as desired.

2. By 1. we know that the left-hand side of this equality is the size of the collection of all subsets of $\{1, \ldots, n\}$. But, this is also true of the right-hand side. Indeed, one may count the number of subsets of $\{1, \ldots, n\}$ by counting the number of subsets having k elements for $k = 0, \ldots, n$. But, by definition, the number of subsets having k elements in $\binom{n}{k}$. As summing over k gives the right-hand side of our desired equality, we are done.

Rubric:

1.
   - **(7 pts)** Coherence in proof writing.
   - **(2 pts)** Setting up induction correctly.
   - **(3 pts)** Precise argument for base case.
   - **(8 pts)** Correct execution of inductive step.
2.
   - **(3 pts)** Coherence in proof writing.
   - **(3 pts)** Correct strategy for how to use 1. to prove equality.
   - **(3 pts)** Correct execution of strategy.